Some Introductory Remarks on Bayesian Inference

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Seminar on Bayes Theory, TU Berlin, SS07

Overview

1. Introduction
   - Bayes Rule
2. Conjugacy
   - Example: Bernoulli distribution
   - Example: Gaussian random variables
   - Exponential Families
3. Philosophical Background
   - On Interpretations of Probability Theory
   - Bayesianism vs. Frequentism in Terms of Modelling

Bayes Rule

Ingredients:

- Model M
- Data D
- Prior $P(M)$
- Conditional Probability $P(D|M)$
- Bayes Rule $P(M|D) = \dfrac{P(D|M)P(M)}{P(D)} = \dfrac{P(D|M)P(M)}{\int P(D|M)P(M)dM}$

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$\rightsquigarrow$ multiple data points by independence assumption:

$$P(D_1, \ldots, D_n|M) = \prod_{i=1}^{n} P(D_i|M).$$

An example

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So what is $P(\text{evolution}|\text{the bible})$?

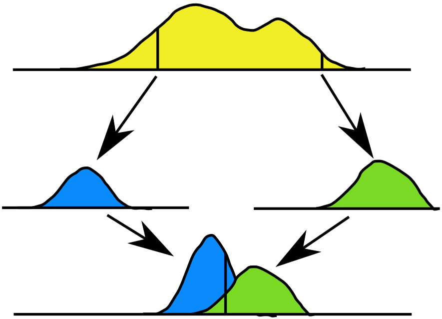
Or $P(\text{intelligent design}|\text{fossils})$?

Or $P(\text{the Matrix}|\text{fossils})$ vs. $P(\text{the Matrix}|\text{many déjà vus})$?

Alternatively...

$\int P(D|M)P(M)dM$ looks like one step in a Markov chain.

$\rightsquigarrow$ models are weighted according to their contribution to D



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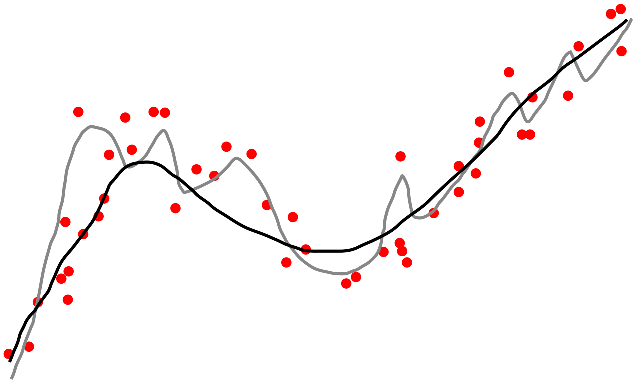


Conjugacy

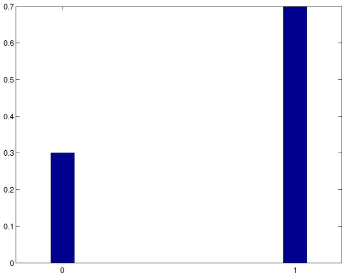
Depending on the probabilities involved, computing Bayes formular requires one integration which may be infeasible.

However, for many probability distributions, it is possible to choose a prior such that

- Bayes rule can be applied exactly,
- the posterior has the same functional form as the prior.

This is called *conjugacy*, and the prior is called the *conjugate prior*.

Bernoulli distribution



$$P(x = 1|\mu) = \mu \qquad P(x = 0|\mu) = 1 - \mu$$

$$\rightsquigarrow P(x|\mu) = \mu^x (1-\mu)^{1-x}$$

Guessing the prior

$$P(M|D) = \frac{PD|MP(M)}{P(D)} \propto P(D|M)P(M).$$

Approach: Forget the normalization, look for a $P(M|\theta)$ such that

$$P(D|M)P(M|\theta) \propto P(M|\theta')$$

For example: $\mu^a(1-\mu)^b$:

$$\mu^x(1-\mu)^{1-x}\mu^a(1-\mu)^b = \mu^{x+a}(1-\mu)^{b+1-x}$$

Finding the normalization

Fortunately, this has already been carried out and the correct prior distributions can be found (somewhere)...

Beta distribution:

$$\mathrm{Beta}(\mu|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\mu^{a-1}(1-\mu)^{b-1}.$$

Expectation: $a/a+b$

($\Gamma(n)$ interpolates the factorial, $\Gamma(n) = (n-1)!$).

Interpreting the prior: Pseudo-counts

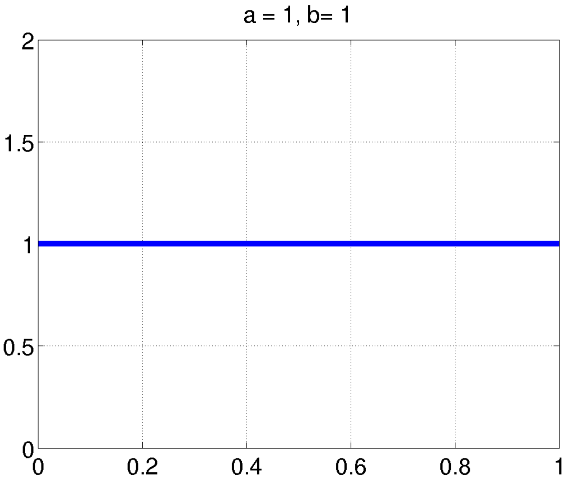
a, b are "pseudo-counts":

$$\mu^x(1-\mu)^{1-x}\mu^{a-1}(1-\mu)^{b-1} = \mu^{a+x-1}(1-\mu)^{b+(1-x)-1}$$

Therefore:

$$\begin{aligned} a &\to a+1 \quad \text{when} \quad x = 1 \\ b &\to b+1 \quad \text{when} \quad x = 0 \end{aligned}$$

The Beta-Distribution



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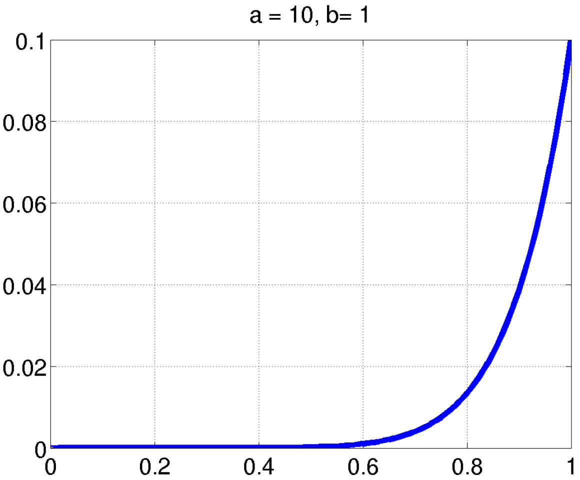

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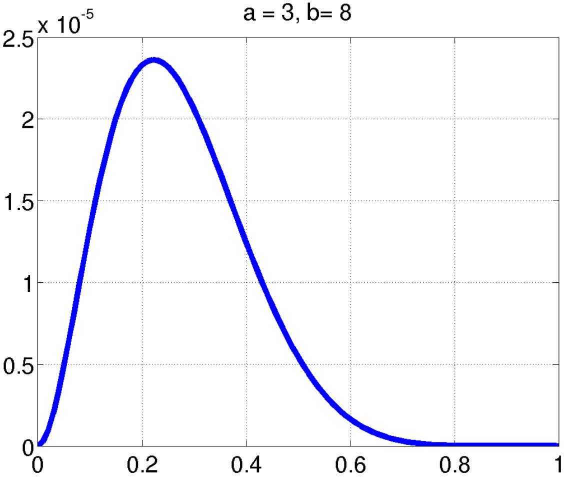

In a similar manner...

Binomial distribution:

$$\binom{n}{k}\mu^k(a-\mu)^{n-k} \quad \Longrightarrow \quad \mathrm{Beta}(\mu|a,b).$$

Multionimal distribution:

$$\binom{n}{n_1 n_2 \dots n_K}\prod_{k=1}^{K}\mu_k^{n_k} \quad \Longrightarrow \quad \text{Dirichlet distribution}$$

$$\mathrm{Dir}(\mu|\alpha) = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_K)}\prod_{k=1}^{K}\mu_k^{\alpha-1}.$$

The Gaussian

The Gaussian distribution:

$$p(x|\mu, \sigma^2) \propto e^{-(x-\mu)^2/2\sigma^2}$$

Let us guess the correct prior for μ: it should be a quadratic function x:

$$p(\mu|a, b) \propto e^{-a(x-b)^2}$$

... which is basically again a Gaussian distribution.

Posterior for n data points:

$$\mu_n = \frac{\sigma^2}{n\sigma_0^2 + \sigma^2}\mu_0 + \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2}\mu_{ML}$$
$$\frac{1}{\sigma_n^2} = \frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}.$$

The Gaussian

Prior for σ^2: Rewrite $\lambda = 1/\sigma^2$, then

$$p(x|\mu, \lambda) \propto \lambda^{1/2} e^{-\lambda(x-\mu)^2/2}.$$

Guessing the prior:

$$\rightsquigarrow \qquad \lambda^b e^{-b\lambda}$$

This leads to the Gamma-distribution:

$$\Gamma(\lambda|a, b) = \frac{1}{\Gamma(a)} b^a \lambda^{a-1} e^{-b\lambda}.$$

Posterior for n data points:

$$a_N = a_0 + \frac{n}{2}$$
$$b_N = b_0 + \frac{n}{2}\sigma^2_{ML}.$$

Exponential Families

In general, conjugate priors exist for distributions from the *exponential family*.

$$p(x|\theta) = h(x)e^{\langle \theta, x \rangle - \psi(\theta)}.$$

Guessing the prior...

$$p(\theta|a, b) \propto e^{\langle \theta, a \rangle - b\psi(\theta)}.$$

Because:

$$e^{\langle \theta, x \rangle - \psi(\theta)} e^{\langle \theta, a \rangle - b\psi(\theta)} = e^{\langle \theta, a+x \rangle - (b+1)\psi(\theta)}$$

Exponential Families (cont'd)

Likelihood	Prior/Posterior
Gaussian (mean)	Gaussian
Gaussian (variance)	Gamma
Poisson	Gamma
Gamma	Gamma
Binomial	Beta
Negative Binomial	Beta
Multinomial	Dirichlet

Bayesianism vs. Frequentism

Frequentism: Maximum-likelihood, Hypothesis Testing, Unbiased Estimates, Support Vector Machines, etc.

Bayesianism: Bayesian estimation, Gaussian Processes, Belief Networks, Factor Graphs, etc.

Irreconcilable Differences or Two Sides of the Same Coin?

Interpretations of Probability Theory

P-Theo does not provide any linkage to the world.

It's basically just this:

$$P(\emptyset) = 0, \quad P(\overline{A}) = 1 - P(A), \quad P(\dot{\bigcup}_i A_i) = \sum_i P(A_i)$$

$$E(X), \quad P(A \cap B) = P(A)P(B) \quad P(A|B) = P(A \cap B)/P(B)$$

From this, everything else is derived, including laws of large numbers, etc.

Bayesianism vs. Frequentism

Use P-Theo to...

Frequentism: ... model independent repeatable experiments

$\rightsquigarrow$ if I sum up many realizations, they will be close to the expectation.

Bayesianism: ... model computations on belief distributions

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Except for: Bayesian approaches result in posterior distribution while Frequentist methods usually just return a single solution.

B vs F—in terms of modelling

Machine learning methods can roughly be decomposed in terms of

- Modelling (what is it I want to learn)
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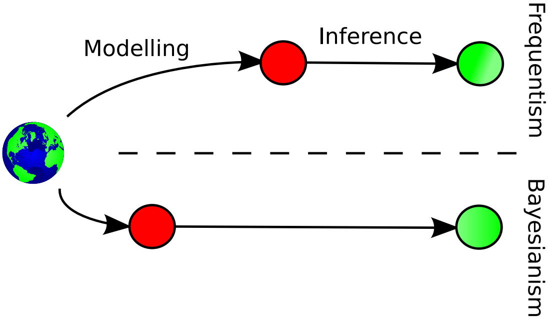
And this holds for both:

	Bayesians	Frequentist
modelling	$P(D\|M)$	loss function
regularization	$P(M)$	regularization
inference	Bayes-rule	optimization

B vs F—different kinds of uncertainty

Frequentism: modelling is kind of inexact, but at least inference is exact.

Bayesianism: modelling is clear, but inference is kind of inexact.



B vs F—irreconcilable differences?

Maybe, since tools are very different:

Frequentist: know which computations on samples converge/concentrate, optimization theory (convex optimization, gradient descent, interior point methods...), etc.

Bayesians: probability distributions, which priors make sensible computations, sampling methods like MCMC, approximation methods.

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At least: You don't have to choose! You can learn both. And of course, you can combine both ;)

Summary

- Bayes rule
- Conjugate priors
- Bayesianism and Frequentism